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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS [1]

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Abstract

In this paper, the initial value problem is discussed for a system of fractional differential equations and new criteria on existence and uniqueness of solutions are obtained.

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1. Introduction

In the recent years, many works have been devoted to the study of initial value problems for fractional differential equations. See, for example papers [1] – [10] and the references therein. In fact, the differential equations involving differential operators of fractional order $0 < \alpha < 1$, appear to be important in modelling several physical phenomena [11] – [19].

In this paper, we study the initial value problem (IVP) for a system of fractional differential equations

$$\begin{cases} D^\alpha x(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}, \tag{1}$$

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where the fractional derivative is in the sense of Caputo's definition, the function $f(t,x) : R \times R^n \to R^n$ is called vector field, and the dimension $n \geq 1$. Particularly, R^n endowed a proper norm $||\cdot||$ becomes a complete metric space.

Caputo's derivative of order $\alpha \in (0,1)$ and with the lower limit t_0 is defined by

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} x'(s) ds.$$

Let

$$J = [t_0 - a, t_0 + a], B = \{x \in R^n | \; ||x - x_0|| \leq b\},$$

$$E = \{(t,x) \in R \times R^n | \;\; t \in J, x \in B\}.$$

A function $x \in C^1(J, R^n)$ is said to be a solution of (1) if x satisfies the equation $D^\alpha x(t) = f(t, x(t))$ a.e. J, and the condition $x(t_0) = x_0$.

In 2008, Lakshmikantham and Vatsala [3] give the following existence result for IVP (1).

Theorem A. ([3]) *Assume that* $f \in C([t_0, t_0 + a] \times B_1, R)$ *and let* $|f(t,x)| \leq M$ *on* $[t_0, t_0 + a] \times B_1$, *where* $B_1 = \{x \in R | \; |x - x_0| \leq b\}$. *Then the IVP (1) possesses at least one solution* $x(t)$ *on* $[t_0, t_0 + h]$, *where* $h = \min\{a, [\frac{b}{M}\Gamma(\alpha+1)]^{1/\alpha}\}$, $0 < \alpha < 1$.

In 2007, Lin [4] obtained the following local existence results for IVP (1).

Theorem B. ([4]) *Assume that the function* $f : E \to R^n$ *satisfies the following conditions:*

(H$_1$*)* $f(t,x)$ *is Lebesgue measurable with respect to* t *on* J*;*

(H$_2$*)* $f(t,x)$ *is continuous with respect to* x *on* B*;*

(H$_3$*) there exists a real-valued function* $m(t) \in L^2(J)$ *such that*

$||f(t,x)|| \leq m(t)$, *for almost every* $t \in J$ *and all* $x \in B$.

Then, for $\frac{1}{2} < \alpha < 1$, *there at least exists a solution of the IVP (1) on the interval* $[t_0 - h, t_0 + h]$ *for some positive number* h.

Theorem C. ([4]) *All the assumptions of Theorem B hold. Assume that (H*$_4$*) there exists a real-valued function* $\mu(t) \in L^4(J)$ *such that*

$||f(t,x) - f(t,y)|| \leq \mu(t)||x - y||$, *for almost every* $t \in J$ *and all* $x, y \in B$.

Then, for $\frac{1}{2} < \alpha < 1$, *there exists a unique solution of the IVP (1) on* $[t_0 - h, t_0 + h]$ *with some positive number* h.

In Remark 2.3 of [4], it is mentioned that Theorems B and C could be generalized to the case where $\alpha \in (0, \frac{1}{2})$ provided that $m(t)$ is bounded on $[t_0 - a, t_0 + a]$.

In this paper, by using some different methods and new techniques, we obtain various criteria on existence and uniqueness of solutions for IVP (1). Our results improve/extend Theorems A, B and C. In particular, we remove the restrictive conditions that $||f(t,x)|| \leq M$ in Theorem A and that $\alpha > \frac{1}{2}$ in Theorems B and C, and relax the hypothesis (H_3) and (H_4).

3. Main Results

THEOREM 1. *Assume that the function $f : E \to R^n$ satisfies the following conditions of Carathéodory type:*

(i) $f(t,x)$ is Lebesgue measurable with respect to t on J;

(ii) $f(t,x)$ is continuous with respect to x on B;

(iii) there exist a constant $\beta \in (0, \alpha)$ and a real-valued function $m(t) \in \mathrm{L}^{\frac{1}{\beta}}(J)$ such that $||f(t,x)|| \leq m(t)$, for almost every $t \in J$ and all $x \in B$.

Then, for $\alpha \in (0,1)$, there at least exists a solution of the IVP (1) on the interval $[t_0 - h, t_0 + h]$, where $h = \min\{a, [\frac{b\Gamma(\alpha)}{M}(\frac{\alpha-\beta}{1-\beta})^{1-\beta}]^{\frac{1}{\alpha-\beta}}\}$ and $M = (\int_{t_0}^{t_0+a}(m(s))^{\frac{1}{\beta}}ds)^{\beta}$.

P r o o f. The IVP (1) for the case where $t \in [t_0, t_0+h]$ are only discussed. Similar approach could be used to verify the case where $t \in [t_0 - h, t_0]$. We know that $f(t, x(t))$ is Lebesgue measurable in $[t_0, t_0 + h]$ according to the conditions (i) and (ii). Direct calculation gives that $(t-s)^{\alpha-1} \in L^{\frac{1}{1-\beta}}[t_0, t]$, for $t \in [t_0, t_0 + h]$. In light of the Hölder inequality, we obtain that $(t-s)^{\alpha-1}f(s, x(s))$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in [t_0, t_0 + h]$, and

$$\int_{t_0}^{t} \|(t-s)^{\alpha-1}f(s,x(s))\|ds \leq \left(\int_{t_0}^{t}((t-s)^{\alpha-1})^{\frac{1}{1-\beta}}ds\right)^{1-\beta}\left(\int_{t_0}^{t}(m(s))^{\frac{1}{\beta}}ds\right)^{\beta}.$$

For $x \in C([t_0, t_0 + h], R^n)$, define norm of $\|x\|_* = \sup_{s\in[t_0,t_0+h]} ||x(s)||$. Then $C([t_0, t_0 + h], R^n)$ with $\|\cdot\|_*$ is a Banach space. Let $\Omega = \{x \in C([t_0, t_0 + h], R^n) : \ ||x - x_0||_* \leq b\}$. Then, Ω is a closed, bounded and convex subset of $C([t_0, t_0 + h], R^n)$.

We now define a mapping on Ω as follows. For an element $x \in \Omega$, let

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds, \qquad t \in [t_0, t_0 + h].$$

a) We shall show that for any $x \in \Omega$, $Tx \in \Omega$.

In fact, by using the Hölder inequality and the condition (iii), we get

$$\begin{aligned}
||(Tx)(t) - x_0|| &\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} m(s) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t} ((t-s)^{\alpha-1})^{\frac{1}{1-\beta}} ds \right)^{1-\beta} \left(\int_{t_0}^{t} (m(s))^{\frac{1}{\beta}} ds \right)^{\beta} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} (t-t_0)^{\alpha-\beta} \left(\int_{t_0}^{t_0+h} (m(s))^{\frac{1}{\beta}} ds \right)^{\beta} \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{1-\beta}{\alpha-\beta} \right)^{1-\beta} h^{\alpha-\beta} M \\
&\leq b, \quad \text{for} \quad t \in [t_0, t_0 + h].
\end{aligned}$$

This means that $||Tx - x_0||_* \leq b$. Hence, T is a mapping from Ω into itself.

b) We now show that T is completely continuous.

First, we will show that T is continuous. For any $x_m, x \in \Omega, m = 1, 2, ...$ with $\lim_{m\to\infty} ||x_m - x||_* = 0$, we get

$$\lim_{m\to\infty} x_m(t) = x(t), \quad \text{for} \quad t \in [t_0, t_0 + h].$$

Thus, by the condition (ii), we have

$$\lim_{m\to\infty} f(t, x_m(t)) = f(t, x(t)), \quad \text{for} \quad t \in [t_0, t_0 + h].$$

So, we can conclude that

$$\sup_{s\in[t_0,t_0+h]} ||f(s, x_m(s)) - f(s, x(s))|| \to 0, \quad \text{as} \quad m \to \infty.$$

On the other hand,

$$\begin{aligned}
||(Tx_m)(t) - T(x)(t)|| &= \frac{1}{\Gamma(\alpha)} \left\| \int_{t_0}^{t} (t-s)^{\alpha-1} [f(s, x_m(s)) - f(s, x(s))] ds \right\| \\
&\leq \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sup_{s\in[t_0,t_0+h]} ||f(s, x_m(s)) - f(s, x(s))||,
\end{aligned}$$

which implies

$$||Tx_m - Tx||_* \leq \frac{h^\alpha}{\Gamma(\alpha+1)} \sup_{s\in[t_0,t_0+h]} ||f(s,x_m(s)) - f(s,x(s))||.$$

Hence,

$$||Tx_m - Tx||_* \to 0 \text{ as } m \to \infty.$$

This means that T is continuous.

Next, we show $T\Omega$ is relatively compact. It suffices to show that the family of functions $\{Tx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, t_0 + h]$.

For $x \in \Omega$, we get

$$\|Tx\|_* \leq \|x_0\| + b,$$

which means that $\{Tx : x \in \Omega\}$ is uniformly bounded.

On the other hand, for any $t_1, t_2 \in [t_0, t_0 + h], t_1 < t_2$, by using the Hölder inequality, we have

$$\begin{aligned}
&||(Tx)(t_2) - (Tx)(t_1)|| \\
&= \frac{1}{\Gamma(\alpha)}\left\| \int_{t_0}^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) f(s,x(s))ds \right. \\
&\left. + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s,x(s))ds \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} ||((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) f(s,x(s))||ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} ||(t_2-s)^{\alpha-1} f(s,x(s))||ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}) m(s)ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} m(s)ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_{t_0}^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})^{\frac{1}{1-\beta}} ds \right)^{1-\beta} \left(\int_{t_0}^{t_1} (m(s))^{\frac{1}{\beta}} ds \right)^{\beta}
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}((t_2-s)^{\alpha-1})^{\frac{1}{1-\beta}}ds\right)^{1-\beta}\left(\int_{t_1}^{t_2}(m(s))^{\frac{1}{\beta}}ds\right)^{\beta}$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t_1}((t_1-s)^{\frac{\alpha-1}{1-\beta}}-(t_2-s)^{\frac{\alpha-1}{1-\beta}})ds\right)^{1-\beta}\left(\int_{t_0}^{t_0+h}(m(s))^{\frac{1}{\beta}}ds\right)^{\beta}$$

$$+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta}\left(\int_{t_0}^{t_0+h}(m(s))^{\frac{1}{\beta}}ds\right)^{\beta}$$

$$\le \frac{M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\Big((t_1-t_0)^{\frac{\alpha-1}{1-\beta}+1}+(t_2-t_1)^{\frac{\alpha-1}{1-\beta}+1}$$

$$-(t_2-t_0)^{\frac{\alpha-1}{1-\beta}+1}\Big)^{1-\beta}+\frac{M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\left((t_2-t_1)^{\frac{\alpha-1}{1-\beta}+1}\right)^{1-\beta}$$

$$\le \frac{2M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}(t_2-t_1)^{\alpha-\beta}.$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. Therefore $\{Tx : x \in \Omega\}$ is equicontinuous on $[t_0, t_0+h]$, and hence $T\Omega$ is relatively compact. By Schauder's fixed point theorem, there is a $x^* \in \Omega$ such that $Tx^* = x^*$, that is

$$x^*(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x^*(s))ds, \qquad t \in [t_0, t_0+h].$$

It is easy to see that x^* is a solution of IVP (1) on $[t_0, t_0+h]$. This completes the proof. ∎

Corollary 1. *Let $f \in C(J \times B, R^n)$. Then, for $\alpha \in (0,1)$, there at least exists a solution of the IVP (1) on the interval $[t_0 - h', t_0 + h']$, where $h' = \min\{a, [\frac{b}{M'}\Gamma(\alpha+1)]^{\frac{1}{\alpha}}\}$ and $M' = \sup_{(t,x)\in J\times B} f(t,x)$.*

The proof of Corollary 1 is similar to that of Theorem 1, it is therefore omitted.

Remark 1. Theorem 1 improves essentially Theorem B by removing the restrictive condition that $\alpha > \frac{1}{2}$, and relaxing the condition (H_3). Corollary 1 extends Theorem A.

Theorem 2. *All the assumptions of Theorem 1 hold. Assume that* *(iv) there exist constant* $\gamma \in (0, \alpha)$ *and a real-valued function* $\mu(t) \in L^{\frac{1}{\gamma}}[t_0, t_0 + a]$ *such that* $||f(t, x) - f(t, y)|| \leq \mu(t)||x - y||$, *for almost every* $t \in J$ *and all* $x, y \in B$.

Then there exists a unique solution of IVP (1) on $[t_0 - h_1, t_0 + h_1]$, *where* $h_1 < \min\left\{a, h, \left[\frac{\Gamma(\alpha)}{M_1}(\frac{\alpha-\gamma}{1-\gamma})^{1-\gamma}\right]^{\frac{1}{\alpha-\gamma}}\right\}$ *and* $M_1 = \left(\int_{t_0}^{t_0+a}(\mu(s))^{\frac{1}{\gamma}}ds\right)^{\gamma}$.

P r o o f. Let $\Omega = \{x \in C([t_0, t_0 + h_1], R^n) : \ ||x - x_0||_* \leq b\}$. For an element $x \in \Omega$, let

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s, x(s))ds, \qquad t \in [t_0, t_0 + h_1].$$

Similarly, we can prove that $T\Omega \subseteq \Omega$ and $(Tx)(t)$ is continuous on $[t_0, t_0 + h_1]$. Now we shall show that operator T is a contraction operator on Ω.

In fact, for $x, y \in \Omega$, we have

$$\begin{aligned}
&||(Tx)(t) - (Ty)(t)|| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-s)^{\alpha-1}\mu(s)ds\right)\max_{s\in[t_0,t_0+h_1]}||x(s) - y(s)|| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-s)^{\alpha-1}\mu(s)ds\right)||x - y||_* \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}((t-s)^{\alpha-1})^{\frac{1}{1-\gamma}}ds\right)^{1-\gamma}\left(\int_{t_0}^{t}(\mu(s))^{\frac{1}{\gamma}}ds\right)^{\gamma}||x - y||_* \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma}(t-t_0)^{\alpha-\gamma}\left(\int_{t_0}^{t_0+h_1}(\mu(s))^{\frac{1}{\gamma}}ds\right)^{\gamma}||x - y||_* \\
\leq\ & \frac{M_1}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma}h_1^{\alpha-\gamma}||x - y||_*, \quad \text{for} \ \ t \in [t_0, t_0 + h_1].
\end{aligned}$$

This implies that

$$||Tx - Ty||_* \leq c||x - y||_*$$

where $c = \frac{M_1}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma}h_1^{\alpha-\gamma} \in (0, 1)$, which proves that T is a contraction mapping. In view of the contraction mapping principle, T has the unique fixed point x, which is obviously a solution of IVP (1) on $[t_0, t_0+h_1]$. The proof is complete. ■

Remark 2. Theorem 2 improves essentially Theorem C by removing the restrictive condition that $\alpha > \frac{1}{2}$, and relaxing the condition (H_4).

Corollary 2. *Let $f \in C(J \times B, R^n)$. Assume that there exists a constant $L > 0$ such that $||f(t,x) - f(t,y)|| \le L||x-y||$, for almost every $t \in J$ and all $x, y \in B$. Then, for $\alpha \in (0,1)$, there exists a unique solution of IVP (1) on the interval $[t_0 - h_1', t_0 + h_1']$, where $h_1' < \min\{a, h', [\frac{1}{L}\Gamma(\alpha+1)]^{\frac{1}{\alpha}}\}$.*

The proof of Corollary 2 is similar to that of Theorem 2, it is therefore omitted.

Theorem 3. *Assume that the condition (iv) of Theorem 2 holds. If the solutions of IVP (1) exist on $[t_0 - h, t_0 + h]$, then the solution of IVP (1) is unique, where $h < \min\{a, [\frac{\Gamma(\alpha)}{M_1}(\frac{\alpha-\gamma}{1-\gamma})^{1-\gamma}]^{\frac{1}{\alpha-\gamma}}\}$.*

P r o o f. Assume that $y(t)$ and $x(t)$ are the solutions of IVP (1) on $[t_0, t_0 + h]$. Then

$$y(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, y(s)) ds, \qquad t \in [t_0, t_0 + h]$$

and

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x(s)) ds, \qquad t \in [t_0, t_0 + h].$$

Hence

$$||y(t)-x(t)|| = \frac{1}{\Gamma(\alpha)} \left\| \int_{t_0}^{t} (t-s)^{\alpha-1} [f(s,y(s)) - f(s,x(s))] ds \right\|, \quad t \in [t_0, t_0+h].$$

By (iv), we have

$$||y(t) - x(t)|| \le c \max_{s \in [t_0, t_0+\delta]} ||y(s) - x(s)||, \qquad t \in [t_0, t_0 + h],$$

where $c = \frac{M_1}{\Gamma(\alpha)}(\frac{1-\gamma}{\alpha-\gamma})^{1-\gamma} h^{\alpha-\gamma} \in (0,1)$. Therefore

$$\max_{s \in [t_0, t_0+h]} ||y(t) - x(t)|| \le c \max_{s \in [t_0, t_0+h]} ||y(s) - x(s)||, \qquad t \in [t_0, t_0 + h]$$

which implies

$$x(t) \equiv y(t), \qquad t \in [t_0, t_0 + h].$$

The proof is complete. ∎

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